REFERENCES

1. CHAPLYGIN S.A., On the theory of motion of non-holonomic systems. In: Chaplygin, S.A., Analysis of the dynamics of non-holonomic systems (Issledovaniya po dynamike negolonomnykh sistem), Gostekhizdat, Moscow-Leningrad, 1949.
2. ILIEV IL., The S.A. Chaplygin reducing factor, Teor. i Pril. Mekh. Vol.10, No.1, 1980.
3. NORDEN A.P., Spaces of an affine connectedness (Prostranstva affinnoi svyaznosti) Gostekhizdat, Moscow-Leningrad, 1950.
4. FUFAEV N.A., The Chaplygin equations and a theorem on the reducing factor in quasicoordinates, Prikl Matem. Mekhan., Vol.25, No.3, 1961.
5. SHALAEV V.G., On Chaplygin's theorem in non-holonomic coordinates, Nauch. tr. Tashkent un-ta, No.242, 1964.
6. SHUL'GIN M.F., On the dynamic Chaplygin equations in the case of conventional non-integrable equations, Prikl. Matem. Mekhan., Vol.18, No.6, 1954.
7. ILIEV IL., Another form of the equations in permissible vectors, Nauch. Trud. na Vys. Ped. Inst. Vol.8, No.2, 1970.
8. ILIEV IL. The equivalence of non-holonomic systems, Nauch. Trud. na Plovdivskii Univ. Vol.13, No.1, 1975.
9. ILIEV IL. Other properties on the equivalence of mechanical systems, Nauch, Trud. na Plovdivskii Univ. Vol.13, No.1, 1975.
10. NOVOSELOV V.S., Application of the Helmholtz method to study the motion of non-holonomic systems, Vestn. LGU, Vol.1, No.1, 1958.
11. NOVOSELOV V.S., Application of the Helmholtz method to study the motion of Chaplygin systems. Vestn. LGU, Vol.3, No.13, 1958.

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FLOW OF A MULTILAYER IDEAL INCOMPRESSIBLE AND HEAVY FLUID PAST A BODY*

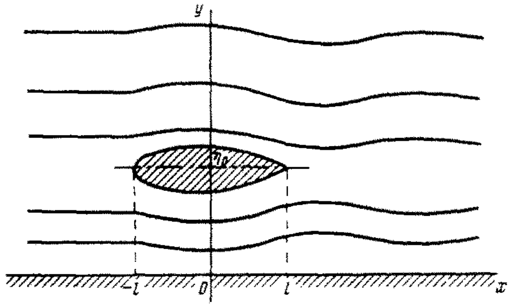
K.A. BEZHANOV and A.M. TER-KRIKOROV

The two-dimensional steady flow of a layered fluid past a body with discontinuous stratification is disucssed. The number of layers in finite, and the channel which has a horizontal floor is open. To study the flow behind the body, a hypothesis on the possibility of approximating the velocity profile at the body boundary by that which arises in weightless flow (see /1,2/) is postulated. A boundary value problem for a second-order elliptic equation in combined Euler-Lagrange variables is formulated. The problem is formulated in a rectilinear band with a separation, and under the conditions of consistency, on a finite number of parallel straight lines which correspond to the separation boundary. The introduction of a measure which gives rise to a monotonic density distribution in a non-perturbed flow, makes it possible to reduce the boundary value problem to the symmetrization of Fredholm-type kernels. The linearized equation is solved by Fourier methods.

The results obtained in /3/ are amplified: it is shown that for any specified Froude number, the corresponding homogeneous integral equation has only a finite number of positive eigenvalues to which the oscillation modes correspond. It is also shown that if the flow velocity is close to one of a denumerable set of propagation velocities of long-wave modes, the corresponding harmonic becomes stronger because of the resonance.

1. Formulation of the problem. Consider the two-dimensional steady flow of an ideal incompressible heavy stratified fluid past a body T_0: $(|x| \leqslant l,\ y_-(x) \leqslant y \leqslant y_+(x))$, where $y_+(x)$ and $y_-(x)$ are known functions which define the body shape. The Ox axis is directed along the horizontal floor of the channel, and the Oy axis runs vertically upwards (see the figure). At the boundaries $y_k(x)$ of the layer T_k, the density ρ and the tangential component of the velocity $\mathbf{V}$ suffer a discontinuity, and the pressure p and the normal

*Prikl. Matem. Mekhan., 49,3,392-400,1985 component of the velocity vector are continuous, T_k: $(-\infty < x < +\infty,\ y_{k-1}(x) < y < y_k(x))$, $k = 1, 2, \ldots, n$.



For the set of domains T (see /3/), the boundary value problem for the modified stream function has the form

$$\nabla^2\psi - gy\rho'(\psi) = \Phi'(\psi) \tag{1.1}$$
$$\left[\frac{1}{2}(\nabla\psi)^2 + gy\rho(\psi) - \Phi(\psi)\right]_k(x) = 0, \quad k = 1, 2, \ldots, n$$
$$[\psi]_k(x) = 0,\ \psi(x, y_k(x)) = \psi_k,\ k = 1, 2, \ldots, n$$
$$\psi(x, 0) = 0,\ \psi(x, y_\pm(x)) = \psi_0$$
$$\Phi(\psi) = \frac{1}{2}\rho(\psi)\mathbf{V}^2 + p + gy\rho(\psi), \quad T = \bigcup_{k=1}^{n} T_k \setminus T_0$$

Here and below, a prime denotes differentiation with respect to the corresponding argument, $[f]_k(x)$ is the jump of the function $f(x, y)$ when it passes through the k-th boundary of discontinuity, the constants ψ_k correspond tothe discontinuity lines, ψ_0 are the stramlines which branch at the body, g, is the acceleration due to gravity, and $\Phi(\psi)$ denotes the Bernoulli function.

As $x \to -\infty$, the one-dimensional unperturbed flow with the parameters

$$\rho = R(\eta), \quad \mathbf{V} = V(\eta)\,\mathbf{i}, \quad p = P(\eta) = g\int_\eta^H (R\,\xi)\,d\xi$$

is given.

Here, η is the Lagrangian coordinate which defines the distance of the unpertrubed streamline from the Ox axis, as $x \to -\infty$, $\mathbf{i}$ is the unit vector of the Ox axis, and H denotes the depth of the unperturbed flow.

It was shown in /3/ that the family of streamlines coincides with the family $\eta(x, y) =$ const, and the density $\rho(\psi) = R(\eta)$ and the Bernoulli function $\Phi(\psi) = \Phi_0(\eta)$ depend on η only, and therefore they are constant on the stramlines.

We will change to dimensionless variables taking the quantity H as the unit of length, and the numbers

$$R_0 = \frac{1}{H}\int_0^H R(\xi)\,d\xi, \quad c = \frac{1}{HR_0}\int_0^H R(\xi)\,V(\xi)\,d\xi$$

as units of the density and velocity.

Let us take x and η as independent, and $y(x, \eta)$ as dependent variables. Then, the band $(-\infty < x < -\infty,\ 0 < \eta < 1)$, corresponds to the region of the flow, the straight line $\eta = 0$ tc the channel floor, the straight line $\eta = 1$ to the free boundary of the flow, the straight lines $\eta = \eta_k$, $k = 1, 2, \ldots, n-1$, $0 < \eta_1 < \ldots < \eta_n = 1$, to the discontinuity boundaries, and the segment G_0: $(|x| \leqslant l,\ \eta = \eta_0)$ to the body. We assume that for $\eta > 1$ we are given a fictitious flow of zero velcoity and ensity.

The function $y(x, \eta)$ satisfies (see /3,4/) the equation

$$\frac{\partial}{\partial\eta}\left(a^2(\eta)\frac{1 + y_x^2}{2y_\eta^2}\right) - a^2(\eta)\frac{\partial}{\partial x}\frac{y_x}{y_\eta} - \nu R'(\eta)\,y = \Phi_0'(\eta) \tag{1.2}$$
$$(\nu = gH/c^2,\ a^2(\eta) = R(\eta)\,V^2(\eta))$$

The subscripts x and η denote differentiation with respect to the corresponding variable, and ν is the reciprocal of the square of the Froude number.

For $|x| \leqslant l$, the boundary conditions on the body have the form

$$y(x, \eta_0 + 0) = \eta_0 + y_+(x), \quad y(x, \eta_0 - 0) = \eta_0 + y_-(x) \tag{1.3}$$

where $y_+(x)$ and $y_-(x)$ are known functions which define the shape of the body, $y_-(x) \leqslant y_+(x)$.

The functions $y(x, \eta)$ and $p(x, \eta)$ are continuous at the boundaries of the layers. This leads to the consistency conditions

$$\left[a^2(\eta)\frac{1 + y_x^2}{2y_\eta^2} - \Phi_0(\eta) - \nu R(\eta)\,y\right]_k(x) = 0, \quad k = 1, 2, \ldots, n \tag{1.4}$$
$$[y]_k(x) = 0, \quad k = 1, 2, \ldots, n$$

The boundary condition on the channel floor, and the asymptotic condition at $-\infty$ have the form

$$y(x,0)=0,\quad \lim_{x\to-\infty} y(x,\eta)=\eta \tag{1.5}$$

The boundary conditions (1.3) greatly complicate the problem. We replace them by a pair of simpler conditions. It follows from (1.3) that

$$[y]_0(x) = \theta(l-|x|)\, y_0(x),\quad y_0(x) = y_+(x) - y_-(x) \tag{1.6}$$

where $\theta(x)$ is the Heaviside unit function, and $[f]_0(x)$ denotes the jump of the function $f(x,\eta)$ in passing through the straight line $\eta=\eta_0$.

We shall express the hypothesis on the possibility of approximating a velocity profile at the body boundary by the profile arising in a flow of a weightless fluid past a body. This hypothesis is justified if the density change in the layer comprising the body is small (a thin body, or a small stratification) /1,2/. For simplicity, we shall consider the case where the body is situated entirely in one layer. Then

$$\left[\frac{1}{2}\rho V^2\right]_0(x)=\left[a^2(\eta)\frac{1+y_x^2}{2y_\eta^2}\right]_0(x)=\theta(l-|x|)\,p_0(x) \tag{1.7}$$
$$p_0(x)=\left[\frac{1}{2}\rho_* V_*^2\right]_0(x)=-[p_*]_0(x)$$

where the parameters of the weightless fluid are marked by asterisks.

Thus, we have Eq.(1.2), and the boundary condition (1.4)-(1.7), as a basis for our discussion.

2. The fundamental integro-differential equation of the flow past a body.

The change

$$y(x,\eta)=\eta+w(x,\eta) \tag{2.1}$$

makes it possible to obtain the following boundary value problem for the aggregate of the domains G:

$$\frac{\partial}{\partial\eta}\left(a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)\right)+a^2(\eta)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+F_2w\right)-\nu R'(\eta)\,w=0 \tag{2.2}$$

$$\left[a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)-\nu R(\eta)\,w\right]_k(x)=0,\quad k=1,2,\ldots,n \tag{2.3}$$

$$\left[a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)\right]_0(x)=\theta(l-|x|)\,p_0(x) \tag{2.4}$$

$$[w]_0(x)=\theta(l-|x|)\,y_0(x),\ w(x,0)=0 \tag{2.5}$$

$$[w]_k(x)=0,\quad k=1,2,\ldots,n,\quad \lim_{x\to-\infty} w(x,\eta)=0 \tag{2.6}$$

$$G=\bigcup_{k=1}^{n}G_k\setminus G_0,\quad G_k\colon(-\infty<x<+\infty,\ \eta_{k-1}<\eta<\eta_k)$$

The expressions for the non-linear operators F_1w and F_2w are given in /3/ but we shall not need them below.

We will reduce problem (2.2)-(2.6) to solving an integro-differential equation (see /3/). For this we integrate (2.2) over the segment $[\eta,1]$, and take advantage of the boundary conditions (2.3) and (2.4):

$$a^2(\eta)\left(\frac{\partial w}{\partial\eta}+F_1w\right)=\nu\int_\eta^1 w(x,\xi)\,d\mu(\xi)-\theta(l-|x|)\,\theta(\eta-\eta_0)\,p_0(x)+\int_\eta^1 a^2(\xi)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}(x,\xi)+F_2w\right)d\xi \tag{2.7}$$

where $\mu(\eta)$ is the Lebesgue-Stieltjes measure generated by the monotonic function $R(\eta)$, and $d\mu(\eta)=-dR(\eta)$.

We divide (2.7) by $a^2(\eta)$, and integrate the result over the segment $[0,\eta]$, using the boundary conditions (2.5) and (2.6). As a result we obtain

$$w(x,\eta)-\nu\int_0^1 G(\eta,\xi)\,w(x,\xi)\,d\mu(\xi)= \tag{2.8}$$
$$\theta(l-|x|)\left(y_0(x)\,\theta(\eta-\eta_0)-p_0(x)\,G(\eta,\eta_0)\right)-$$
$$\int_0^\eta F_1w\,d\xi+\int_0^1 a^2(\xi)\,G(\eta,\xi)\left(\frac{\partial^2w}{\partial x^2}(x,\xi)+\frac{\partial}{\partial x}F_2w\right)d\xi$$
$$G(\eta,\xi)=\begin{cases}q(\eta), & \eta\leqslant\xi\\ q(\xi), & \eta>\xi\end{cases},\quad q(\eta)=\int_0^\eta\frac{d\tau}{a^2(\tau)}$$

If we invert the Fredholm operator on the left-hand side of (2.8), we shall obtain an integro-differential equation which does not contain the measure $\mu(\eta)$ on its left-hand side:

$$w(x,\eta)+\int_0^1 a^2(\xi)\,\Gamma(\eta,\xi,\nu)\,\frac{\partial^2 w}{\partial x^2}(x,\xi)\,d\xi= \tag{2.9}$$
$$\theta(l-|x|)\,(y_0(x)\,\chi_1(\eta,\nu)-p_0(x)\,\chi_2(\eta,\nu))+\Phi w$$
$$\chi_1(\eta,\nu)=\theta(\eta-\eta_0)-\nu\int_{\eta_0}^1 \Gamma(\eta,\xi,\nu)\,d\mu(\xi)$$
$$\chi_2(\eta,\nu)=G(\eta,\eta_0)-\nu\int_0^1 \Gamma(\eta,\xi,\nu)\,G(\xi,\eta_0)\,d\mu(\xi)$$

The expressions for the non-linear operator Φw, and the resolvent $\Gamma(\eta,\xi,\nu)$ are given in /3/. A resolvent has a denumerable set of simple positive poles at the points $\nu=\nu_l$ to which correspond the critical velocities of the long-wave modes.

3. The spectrum of the Fredholm integral equation.

On putting $\Phi w=0$ in (2.9), we obtain an integro-differential equation. To solve it we need to consider an additional Fredholm integral equation with a symmetrizing kernel:

$$z(\eta,\nu)=\lambda\int_0^1 a^2(\xi)\,\Gamma(\eta,\xi,\nu)\,z(\xi,\nu)\,d\xi \tag{3.1}$$

For $\nu\neq\nu_l$, all eigenvalues $\lambda_m\ (m=1,2,\ldots)$ of equation (3.1) are simple and real, and the corresponding eigenfunctions are orthogonal with weight $a^2(\eta)$.

Theorem 1. For a specified value of the parameter ν, the integral equation (3.1) has a finite number of eigenvalues.

Proof. The integral Eq.(3.1) is equivalent to the following boundary value problem in eigenvalues for a second-order differential operator:

$$\frac{d}{d\eta}\left(a^2(\eta)\frac{dz}{d\eta}\right)-\lambda a^2(\eta)\,z-\nu R'(\eta)\,z=0,\ 0<\eta<1 \tag{3.2}$$
$$\left[a^2(\eta)\frac{dz}{d\eta}-\nu R(\eta)\,z\right]_k=0,\quad k=1,2,\ldots,n \tag{3.3}$$
$$z(0,\lambda)=0,\ [z]_k=0,\ k=1,2,\ldots,n-1 \tag{3.4}$$

where $z=z(\eta,\lambda)$ without changing the notation.

The eigenvalues λ_m can be found as follows. First a Cauchy problem for Eq.(3.2) in the segment $[0,\eta_1]$, with the initial conditions

$$z(0,\lambda)=0,\ \frac{dz}{d\eta}(0,\lambda)=\frac{\sqrt{\lambda}}{a(0)} \tag{3.5}$$

is solved when $\eta=0$.

Let this solution be $z_1(\eta,\lambda)$. Then the Cauchy problem for (3.2) is solved in the segment $[\eta_1,\eta_2]$, and the initial conditions for $\eta=\eta_1$, are taken for $k=1$ from the consistency conditions (3.3) and (3.4),

$$z_2(\eta_1,\lambda)=z_1(\eta_1,\lambda)$$
$$\frac{dz_2}{d\eta}(\eta_1,\lambda)=\frac{a^2(\eta_1-0)}{a^2(\eta_1+0)}\frac{dz_1}{d\eta}(\eta_1,\lambda)+\nu\,[R(\eta)]_1\frac{z_1(\eta_1,\lambda)}{a^2(\eta_1+0)}$$

We denote this solution by $z_2(\eta,\lambda)$. Continuing this argument, we can construct the function

$$z=z_k(\eta,\lambda),\ \eta\in[\eta_{k-1},\eta_k],\ k=1,2,\ldots,n$$

which satisfies Eq.(3.2), and all boundary conditions (3.3) and (3.4), with the exception of (3.3) for $k=n$. For λ to be an eigenvalue, and for $z(\eta,\lambda)$ to be a corresponding eigenfunction, it is necessary and sufficient that λ shall be the solution of the equation

$$a^2(1)\frac{dz}{d\eta}(1,\lambda)-\nu R(1)\,z(1,\lambda)=0 \tag{3.6}$$

Thus, the eigenvalues of (3.2) should be the solutions of Eq.(3.6).

If we now assume that the positive eigenvalues of the integral Eq.(3.1) form an infinite sequence, then the boundary point of this sequence should be at infinity. We shall show that this assumption leads to a contradiction.

Let us multiply (3.2) by $z(\eta,\lambda)$, and integrate it over the segment $[0,1]$. Then, using boundary conditions (3.3) and (3.4), we obtain

$$\lambda I_1(\lambda) = \nu I_2(\lambda) - I_3(\lambda)$$

$$I_1(\lambda) = \int_0^1 a^2(\eta) z^2(\eta, \lambda)\, d\eta, \quad I_2(\lambda) = \int_0^1 z^2(\eta, \lambda)\, d\mu(\eta)$$

$$I_3(\lambda) = \int_0^1 a^2(\eta) z_\eta'^2(\eta, \lambda)\, d\eta$$

hence

$$(I_2(\lambda)/I_1(\lambda)) \geqslant \lambda/\nu \tag{3.7}$$

Further, for $0 \leqslant \eta \leqslant 1$, let

$$\alpha \leqslant a(\eta) \leqslant \beta, \ R(\eta) \geqslant \gamma, \ |R'(\eta)| \leqslant \delta \tag{3.8}$$

where α, β, γ and δ are positive constants.

Considering (3.8), for large positive λ we have the following asymptotic form of the solution of Eq.(3.2) (see /5/):

$$z(\eta, \lambda) = \frac{\exp(\sqrt{\lambda}\,\eta)}{a(\eta)}(1 + o(1)), \quad \lambda \to +\infty \tag{3.9}$$

Using the boundary conditions (3.3), (3.4), inequalities (3.8) and formula (3.9), we can obtain by induction the asymptotic solutions for all layers,

$$z_k(\eta, \lambda) = \gamma_k a^{-1}(\eta) \exp(\sqrt{\lambda}\,\eta)\,(1 + o(1)), \ \eta_{k-1} \leqslant \eta \leqslant \eta_k \quad k = 1, 2, \ldots, n \tag{3.10}$$

where the constants γ_k do not depend on λ.

Then for $\lambda_m \to +\infty$, from (3.8) and (3.10) we obtain the expression

$$I_2(\lambda) = \sum_{k=1}^{n} z^2(\eta_k, \lambda) R(\eta_k) - \int_0^1 z^2(\eta, \lambda) R'(\eta)\, d\eta \leqslant C_1 \exp(2\sqrt{\lambda}), \quad C_1 > 0 \tag{3.11}$$

The integral on the right-hand side has been discarded since its order equals $\exp(2\sqrt{\lambda})/\sqrt{\lambda}$.

Similarly, from (3.8) and (3.10) we have

$$I_1(\lambda) \geqslant C_2 \exp(2\sqrt{\lambda})/\sqrt{\lambda}, \ C_2 > 0 \tag{3.12}$$

and from (3.11) and (3.12) there follows the inequality

$$(I_2(\lambda)/I_1(\lambda)) \leqslant C\sqrt{\lambda}, \ C > 0 \tag{3.13}$$

In expressions (3.11)-(3.13), the constants C_1, C_2 and C do not depend on λ.

Since, by assumption, the eigenvalues of the integral equation have a boundary point at infinity, a sequence $\{\lambda_m\}$ exists such that

$$\lim_{m \to \infty} \lambda_m = +\infty, \quad \lambda_m > 0$$

On substituting λ_m into inequalities (3.7) and (3.13), as $\lambda_m \to +\infty$ we obtain

$$(\lambda_m/\nu) \leqslant C\sqrt{\lambda_m}$$

which is impossible. This proves the theorem.

Let us number the eigenvalues of the integral Eq.(3.1) as follows:

$$\lambda_1 > \lambda_2 > \ldots > \lambda_N > 0 > \lambda_{N+1} > \ldots$$

The corresponding eigenfunctions

$$z_m(\eta, \nu), \ m = 1, 2, \ldots, N, N+1, \ldots \tag{3.14}$$

form a full orthonormal system.

For a single-layer model, the assertion of Theorem 1 was obtained earlier in /6/* (*see also the paper by Gorodtsov and Teodorovich, Cherenkov emission of internal waves by uniformly moving sources, Preprint No.183, AN SSSR, Moscow, p.65, 1981). The exact solutions of (1.1) with a finite number of wave harmonics for the case where $\rho'(\psi)$ and $\Phi'(\psi)$ depend linearly on ψ were obtained in /7/.

4. Solution of the integro-differential equation. The integro-differential equation which corresponds to (2.9) is solved by the Fourier method. For this, we expand the functions $\chi_1(\eta, \nu)$ and $\chi_2(\eta, \nu)$ defined by the formulae in (2.9), in series according to system (3.14):

$$\chi_1(\eta, \nu) = \sum_{m=1}^{\infty} \alpha_m z_m(\eta, \nu), \ \chi_2(\eta, \nu) = \sum_{m=1}^{\infty} \beta_m z(\eta, \nu), \ 0 \leqslant \eta \leqslant 1 \tag{4.1}$$

$$\alpha_m = \int_{\eta_0}^{1} a^2(\eta) z_m(\eta, \nu)\, d\eta - \frac{\nu}{\lambda_m} \int_{\eta_0}^{1} z_m(\eta, \nu)\, d\mu(\eta)$$

$$\beta_m = \int_{0}^{1} a^2(\eta) G(\eta, \eta_0) z_m(\eta, \nu)\, d\eta - \frac{\nu}{\lambda_m} \int_{0}^{1} G(\eta, \eta_0) z_m(\eta, \nu)\, d\mu(\eta)$$

We seek the solution in the form

$$w(x, \eta) = \sum_{m=1}^{\infty} B_m(x) z_m(\eta, \nu)$$

where the unknown functions $B_m(x)$ satisfy the equation

$$B_m''(x) + \lambda_m B_m(x) = \lambda_m f_m(x) \qquad (4.2)$$
$$f_m(x) = \theta(l - |x|)\, (\alpha_m y_0(x) - \beta_m p_0(x))$$

The solution of (4.2), which vanishes at $-\infty$ together with its derivative, has the form

$$B_m(x) = B_m^+(x) = \sqrt{\lambda_m} \int_{-\infty}^{x} f_m(\xi) \sin \sqrt{\lambda_m}(x - \xi)\, d\xi, \quad \lambda_m > 0 \qquad (4.3)$$

$$B_m(x) = B_m^-(x) = \frac{\sqrt{|\lambda_m|}}{2} \int_{-\infty}^{+\infty} f_m(\xi)$$
$$\exp(-\sqrt{|\lambda_m|}\, |x - \xi|)\, d\xi, \ \lambda_m < 0$$

Then, using the assertion of Theorem 1, we write the solution of the integro-differential equation as

$$w(x, \eta) = \sum_{m=1}^{N} B_m^+(x) z_m(\eta, \nu) + \sum_{m=N+1}^{\infty} B_m^-(x) z_m(\eta, \nu) \qquad (4.4)$$

It follows from formulae (4.3) and (4.4) that in front of the body the flow is disturbed weakly, and the waves develop downstream at once after the encounter with the body.

To investigate the flow behind the body we write formulae (4.3) and (4.4) for $x > l$,

$$w(x, \eta) = \sum_{m=1}^{N} (M_m \sin \sqrt{\lambda_m}\, x + L_m \cos \sqrt{\lambda_m}\, x) \sqrt{\lambda_m}\, z_m(\eta, \nu) - \frac{1}{2} F(x, \eta, \nu) \qquad (4.5)$$

$$F(x, \eta, \nu) = \sum_{m=N+1}^{\infty} \sqrt{|\lambda_m|}\, z_m(\eta, \nu) \int_{-l}^{l} f_m(\xi) \exp(-\sqrt{|\lambda_m|}(x - \xi))\, d\xi$$

$$M_m = \int_{-l}^{l} f_m(\xi) \cos \sqrt{\lambda_m}\, \xi\, d\xi, \quad L_m = -\int_{-l}^{l} f_m(\xi) \sin \sqrt{\lambda_m}\, \xi\, d\xi \qquad (4.6)$$

Theorem 2. If the functions $y_0(x)$ and $p_0(x)$ are continuous on the segment $[-l, l]$, then the following estimate of the function $F(x, \eta, \nu)$, uniform in x and η, holds for $x > + 2/\sqrt{|\lambda_{N+1}|}$:

$$|F(x, \eta, \nu)| \leqslant C(\nu) \exp(-\sqrt{|\lambda_{N+1}|}\,(x - l)) \qquad (4.7)$$

Proof. It follows from Bessel's inequality that

$$\sum_{m=N+1}^{\infty} \frac{z_m^2(\eta, \nu)}{\lambda_m^2} \leqslant \int_0^1 a^2(\xi) \Gamma^2(\eta, \xi, \nu)\, d\xi \leqslant C_1(\nu) \qquad (4.8)$$

$$\sum_{m=N+1}^{\infty} (\alpha_m^2 + \beta_m^2) \leqslant \int_0^1 a^2(\xi) (\chi_1^2(\xi, \nu) + \chi_2^2(\xi, \nu))\, d\xi = C_2(\nu)$$

$$C_1(\nu) = \max_{0 \leqslant \eta \leqslant 1} \int_0^1 a^2(\xi) \Gamma^2(\eta, \xi, \nu)\, d\xi$$

Since for $x > 0$ we have $|x \exp(-2\sqrt{x})| < 1$,

$$|\lambda_m| \exp\left(-2\sqrt{\left|\frac{\lambda_m}{\lambda_{N+1}}\right|}\right) < |\lambda_{N+1}| \qquad (4.9)$$

Further, let

$$|y_0(x)| \leqslant K, \ |p_0(x)| \leqslant K, \ x > l + 2/\sqrt{|\lambda_{N+1}|} \qquad (4.10)$$

where K is a certain positive constant. Then, using inequalities (4.8)-(4.10), from (4.5) we have

$$|F(x,\eta,\nu)| \leqslant \sum_{m=N+1}^{\infty} 2K(|\alpha_m|+|\beta_m|)|z_m(\eta,\nu)|\exp(-\sqrt{|\lambda_m|}(x-l)) \leqslant$$
$$K|\lambda_{N+1}|\exp(-\sqrt{|\lambda_{N+1}|}(x-l)+2)\sum_{m=N+1}^{\infty}(\alpha_m^2+\beta_m^2+$$
$$2\frac{z_m^2(\eta,\nu)}{\lambda_m^2}\Big) \leqslant C(\nu)\exp(-\sqrt{|\lambda_{N+1}|}(x-l))$$
$$C(\nu) = Ke^2(2C_1(\nu)+C_2(\nu))|\lambda_{N+1}|$$

The theorem is proved.

Theorem 3. When $x \geqslant x_0 > l$, an estimate of the type (4.7) for any derivative of the function $F(x,\eta,\nu)$ holds.

The proof is similar to that of Theorem 2.

The number x_0 depends on the order of the derivative of $F(x,\eta,\nu)$. Thus, when x increases the solution $w(x,\eta)$ becomes smoother, and the peculiarities of the flow caused by small unevenness of the body are smoothed out as $x \to +\infty$.

As $x > l + 2/\sqrt{|\lambda_{N-1}|}$, from formulae (2.1) and (4.5) and Theorem 2 there follows the asymptotic behaviour of the function $y(x,\eta)$, uniform in x and η,

$$y(x,\eta) = \eta + \sum_{m=1}^{N}(M_m \sin\sqrt{\lambda_m}\,x + L_m\cos\sqrt{\lambda_m}\,x)\sqrt{\lambda_m}\,z_m(\eta,\nu) + O(\exp(-\sqrt{|\lambda_{N-1}|}(x-l)) \tag{4.11}$$

(M_m and L_m are defined by (4.6)).

5. Asymptotic behaviour of the solution in the vicinity of the critical values of the parameter ν.

The critical values of the parameter $\nu = \nu_l$, $l = 1, 2, \ldots,$ were defined in /3/ as eigenvalues of the Fredholm equation with the symmetric kernel

$$W(\eta) = \nu\int_0^1 G(\eta,\xi)W(\xi)\,d\mu(\xi) \tag{5.1}$$

The critical velocities of propagation of the long-wave modes correspond to the critical values of ν_l, and the corresponding eigenfunctions $\varphi_l(\eta)$ of Eq.(5.1) determine their amplitudes. Also, the asymptotic form was established as $\nu \to \nu_l + 0$, in this case the l-th term in (4.11) is the main term, and the formulae

$$z_l(\eta,\nu) = \frac{\varphi_l(\eta)}{\sqrt{A_{ll}}} + O(\nu-\nu_l), \quad \lambda_l = \frac{\nu-\nu_l}{A_{ll}} + o(\nu-\nu_l) \tag{5.2}$$
$$\Gamma(\eta,\xi,\nu) = \frac{\varphi_l(\xi)\varphi_l(\eta)}{\nu-\nu_l} + O(1), \quad A_{ll} = \int_0^1 a^2(\eta)\varphi_l^2(\eta)\,d\eta$$

hold.

Substituting (5.2) into (4.1), and leaving only the main terms of the asymptotic forms as $\nu \to \nu_l + 0$, we obtain

$$\alpha_l = -\frac{\nu_l}{\lambda_l\sqrt{A_{ll}}}\int_{\eta_0}^{1}\varphi_l(\eta)\,d\mu(\eta), \quad \beta_l = -\frac{1}{\lambda_l\sqrt{A_{ll}}}\varphi_l(\eta_0) \tag{5.3}$$

Denoting the size of the area occupied by the body by S, and the lifting force due to the velocity circulation by Q, using formulae (1.6) and (1.7) we arrive at the expression

$$S = \int_{-l}^{l} y_0(x)\,dx = \int_{-l}^{l}(y_+(x)-y_-(x))\,dx \tag{5.4}$$
$$Q = \int_{-l}^{l} p_0(x)\,dx = -\int_{-l}^{l}[p_*]_0(x)\,dx = \int_{\partial T} p_*(s)\cos(\boldsymbol{\tau}_0,x)\,ds = -\int_{\partial T} p_*(s)\cos(\mathbf{n}_0,y)\,ds$$

where $\boldsymbol{\tau}_0$ and $\mathbf{n}_0$ are the unit vectors of the tangent and normal to the body boundary ∂T.

Then from (4.2), (4.6) and (5.4) we find

$$M_l = \alpha_l S - \beta_l Q, \; L_l = O(\lambda_l) \tag{5.5}$$

Formula (4.11) becomes noticeably simpler. Using (5.2), (5.3) and (5.5), we obtain the approximate expression

$$y(x,\eta)\approx\eta+\frac{aS-bQ}{\sqrt{\lambda_l}\,A_{ll}}\sin\sqrt{\lambda_l}\,x\varphi_l(\eta) \quad (5.6)$$

$$a=-\nu_l\int_{\eta_0}^{1}\varphi_l(\eta)\,d\mu(\eta),\quad b=-\varphi_l(\eta_0)$$

As $\nu\to\nu_l+0$, a quantity of the order of

$$O(\exp(-\sqrt{|\lambda_{N+1}|}\,(x-l)))+o(\sqrt{(S^2+Q^2)/\lambda_l})$$

is the estimate in (5.6).

For $|x|<l$, series (4.4) converges in the mean-square sense. If we use the asymptotic properties of the eigenvalues λ_m and the eigenfunctions $z_m(\eta,\nu)$ as $m\to+\infty$, then the standard technique of mathematical physics enables us to separate the singularities at the body boundary and to improve the convergence of the series. A more detailed study of the near velocity field would make it possible to determine the distortions to the body shape by the hypothesis of the possibility of approximating the velocity profile at the body boundary by that of a weightless fluid flowing past. This question is not discussed in the present paper. Here we merely remark that in the arrangement discussed the streamline which corresponds to the body remains closed, and the area bounded by this streamline equals the area of the body.

REFERENCES

1. MILES J.W., Internal waves generated by a horizontally moving sourse. Geophys. Fluid Dyn., Vol.1, No.1, 1971.
2. LIGHTHILL J., Waves and fluids./Russian translation/, Mir, Moscow, 1981.
3. BEZHANOV K.A. and TER-KRIKOROV A.M., Multilayer stationary flows of an ideal incompressible fluid over an uneven floor, Prikl. Matem. Mekhan., Vol.48, No.5, 1984.
4. TER-KRIKOROV A.M., On the theory of sustained-type waves in a non-homogeneous fluid, Prikl. Matem. Mekhan., Vol.29, No.3, 1965.
5. FEDORYUK M.V., Asymptotic methods for ordinary differential equations (Asimptoticheskie metody dlya obyknovennykh differentsial'nykh uravnenii), Nauka, Moscow, 1983.
6. DOTSENKO S.F., On the nature of wave motion in a flow when there is an arbitrary density variation with depth, Morskie gidrofiz. issled., No.3, 1973.
7. YIH C.S., Exact solutions for steady two-dimensional flow of a stratified fluid. J. Fluid J. Fluid Mech., Vol.9,No.2,1960.

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ANISOTROPIC TURBULENCE IN THE FLOW OF AN INCOMPRESSIBLE FLUID BETWEEN PARALLEL PLANE WALLS*

V.A. BABKIN

It is shown that in the region adjacent to a solid wall a Newtonian fluid in turbulent flow can be regarded as an oriented Ericsson-Leslie fluid whose defining constants are subject to certain conditions. The logarithmic velocity profile is obtained from the solution found if the molecular viscosity is ignored, when the distance from the wall is small.

1. Consider the confined turbulent flow of an incompressible Newtonian fluid between plane parallel walls in the absence of mass forces. The coordinate system consists of an x-axis directed along the flow, and a y-axis perpendicular to the walls. The wall equation is $y=\pm h$.

The Prandtl semi-empirical theory of the mixing length, and numerous experiments show that in the vicinity of a solid wall the longitudinal averaged velocity u has the following logarithmic profile:

$$\frac{u}{v_*}=\frac{1}{\varkappa}\ln\left(1-\frac{|y|}{h}\right)+C \quad (1.1)$$

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